Nonperturbative approach to the quantum Hall bilayer

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We develop a nonperturbative approach to the quantum Hall bilayer (QHB) at $\nu=1$ using trial wave functions. We predict phases of the QHB for arbitrary distance d and, our approach, in a dual picture, naturally introduces a new kind of quasiparticles—neutral fermions. Neutral fermion is a composite of two merons of the same vorticity and opposite charge. For small d (i.e., in the superfluid phase), neutral fermions appear as dipoles. At larger d dipoles dissociate into the phase of the two decoupled Fermi-liquid-like states. This scenario is relevant for the experimental situation where impurities lock charged merons. In a translation invariant (clean) system, continuous creation and annihilation of meron-antimeron pairs evolves the QHB toward a paired phase. The quantum fluctuations fix the form of the pairing function to $g(z)=1/z^{*}$. A part of the description of the paired phase is the two-dimensional superconductor i.e., BF Chern-Simons theory. The paired phase is not very distinct from the superfluid phase.

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I. INTRODUCTION

The quantum Hall bilayer at $\nu=1$ consists of two layers of two-dimensional (2D) electron gases that are brought close to one another in the quantum Hall regime of strong magnetic fields. When the distance between the layers is much smaller than the average distance between electrons inside each layer, inter- and intra-Coulomb interactions are about the same. Then the expected $\nu=1$ state is the state of a single layer filled lowest Landau level (LLL) generalized to two species. There is obvious degeneracy in dividing electrons into two groups which leads to the phenomenon of spontaneous symmetry breaking[1] and the existence of a Goldstone mode.[2] The expected superfluid behavior was verified also by very large zero-bias voltage peak in tunneling conductance,[3] but no clear evidence was found for finite temperature Berezinskii-Kosterlitz-Thouless (BKT) transition[4] in transport experiments.[5]

Therefore there is a need to systematically address the question of superfluid disordering in the quantum Hall bilayer (QHB). In particular there is a need to understand the role of quantum disordering in this system that becomes important as the distance between the layers is increased. In most of the previous work the starting point for the discussion of the physics of the bilayer was the ground state (GS) for very small distance between the layers as a mean-field solution to which none or some corrections were developed.[4,6] We will take a nonperturbative approach inspired by the Laughlin solution of the $\nu=1/3$ problem in which we will uniquely determine possible wave functions (WFs) for the GSs of the bilayer at an arbitrary distance.

There are two basic paradigms of superfluid disordering that are known: (1) BKT (2D *XY* model) for which the transition proceeds via unbinding of dipoles of vortex-antivortex pairs, and (2) λ transition type [three-dimensional (3D) XY model] for which the transition is characterized by a condensation of vortex-antivortex loops.[7]

On the other hand, in this paper, through an analysis of the allowed possibilities for homogeneous WFs as the distance is varied, we will identify two families of WFs and relate them to the two ways of disordering the QHB superfluid mentioned previously. The families will be introduced in Sec. II.

One family, as it will turn out does not include elementary vortices—merons of QHB, in its description of superfluid disordering. Merons are part of the description of the QHB superfluid for small distances as is well known and well established in Ref. 4. Therefore this family of (homogeneous) WFs can be relevant only for dirty systems—systems with impurities, which can lock merons due to merons being charged quasiparticles. Then the only vortices that may participate in superfluid disordering and on which description of this family of WFs is based are neutral composites of two merons of opposite charges—neutral vortices, and as we will find fermionic quasiparticles that carry only layer degree of freedom. We will show that the superfluid disordering of this family can be understood through a Coulomb (fermionic) plasma picture of dipoles of these neutral fermions. Therefore this family we can consider as the one that exemplifies the BKT way of superfluid disordering, our first paradigm. This whole picture will be corroborated by the fact that the WFs of this family do not incorporate quantum fluctuations (Sec. IV) and, therefore, do not incorporate quantum disordering that is based on merons. The family from the viewpoint of a dual description (i.e., in terms of quasiparticles—neutral fermions) will be analyzed in Sec. III.

The other family incorporates weak pairing among neutral fermions and, as we will show, by assuming a special kind of pairing agrees and correlates with the description of quantum fluctuations of the usual superfluid disordering in a translatory invariant system that one finds in other approaches (field-theoretical). It is expected that this kind of disordering and pairing would lead to a charge-density wave (CDW) solution.[8] Still our general considerations open possibilities for other kinds of weak pairing that can be present in this quantum Hall system. The most likely candidate is the one with pairing function $g(z)\sim\frac{1}{z^{*}}$ that results in nontrivial corrections (from quantum fluctuations and disordering) to the ground-state wave function as the distance is varied. In general we expect that a weak pairing scenario will correspond to the superfluid disordering of the usual superfluid in 2+1 dimension and therefore to the class of 3D *XY*, our second paradigm. The family with weak pairing ansatz will be analyzed in Sec. IV.

With respect to the experiments, where impurities are necessarily present and we can expect also inhomogeneous ground-state solutions, our homogeneous candidates of the first family (without neutral fermion weak pairing) are still possible solutions for which transitions may proceed via dissociation of dipoles—pairs of opposite vorticity neutral fermions. In this sense and as will be more clear later, the quantum phase transitions with respect to changing the distance in Refs. 9–11 correspond to this dissociation. On the other hand, an analysis will show that in a translatory invariant system meron excitations via their loop condensation may produce an intercorrelated paired liquid state for the neutral sector, if a transition to a CDW does not occur.

II. UNIVERSALITY CLASSES OF GROUND STATES

A. Introduction

A great deal is known from the experimental and theoretical point of view of the QHB in the two extremes when the distance between layers, d, is (1) much smaller or (2) much larger than the magnetic length, $l_B=(\hbar/eB)^{1/2}$, where B is the magnetic field, the characteristic distance between the electrons inside any of the layers. When $d \ll l_B$, i.e., inter and intra Coulomb interactions are about the same, the good starting point and description is so-called (111) state,[12]

$$\Psi_{111}(z_\uparrow,z_\downarrow)=\prod_{i<j}(z_{i\uparrow}-z_{j\uparrow})\prod_{k<l}(z_{k\downarrow}-z_{l\downarrow})\prod_{p,q}(z_{p\uparrow}-z_{q\downarrow}), \tag{1}$$

where $z_{i\uparrow}$ and $z_{i\downarrow}$ are two-dimensional complex coordinates of electrons in upper and lower layer, respectively, and we omitted the Gaussian factors. This is suggestive of the exciton binding;[13] any electron coordinate is also zero of the WF for any other electron coordinate—the correlation hole is just opposite to electron. This exciton description can be a viewpoint of the phenomenon of superfluidity found in these systems[2,3] and is closely connected to the concept of composite bosons (CBs) (Ref. 14) that can be used as natural quantum Hall quasiparticles in this system. When $d \gg l_B$ we have the case of the decoupled layers and the GS is a product of single-layer filling factor 1/2 WFs; each describes a Fermi-liquid-like state,[15]

$$\Psi_{1/2}(w)=\mathcal{P}\left\{\mathcal{F}_s(w,\bar{w})\prod_{i<j}(w_{i\uparrow}-w_{j\uparrow})^2\right\}, \tag{2}$$

where $\mathcal{F}_s$ is the Slater determinant of free waves of noninteracting particles in zero magnetic field and $\mathcal{P}$ represents projection to LLL. Underlying quasiparticles are composite fermions (CFs), the usual quasiparticles of the single layer quantum Hall physics.

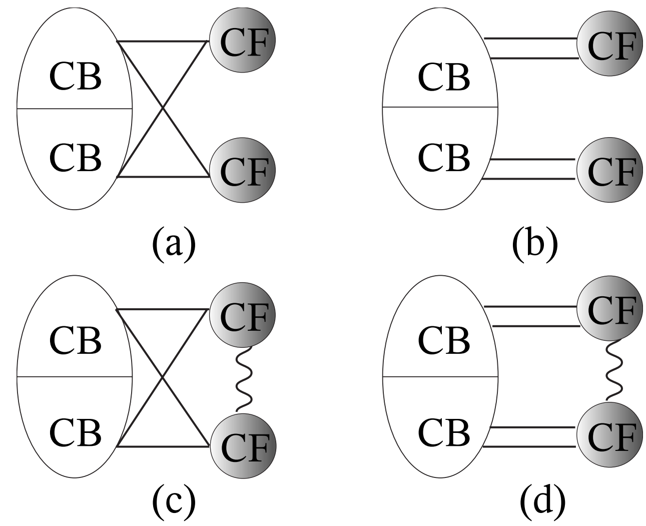

FIG. 1. Universality classes of wave functions.

B. Two families—universality classes of wave functions

To answer the question of intermediate distances we may try to, classically speaking, divide electrons into two groups, one in which electrons correlate as CBs and the other as CFs.[16] The ratio between the numbers of CBs and CFs would be determined by the distance between layers. The WF constructed in this way would need an overall antisymmetrization in the end, but also intercorrelations among the groups as each electron of the system sees the same number of flux quanta through the system (equal to the number of electrons). This requires that the highest power of any electron coordinate is the same as the number of electrons in the thermodynamic limit. If we denote by a line the Laughlin-Jastrow factor $\Pi_{A,B}(z_A-z_B)$ between two groups of electrons, A and B $(A,B=CB,CF)$, the possibilities for the QHB GSWFs can be summarized as in Fig. 1.

If we ignore the possibility of pairing between CFs (Ref. 17) denoted by wriggly lines in Figs. 1(c) and 1(d) we have two basic families of the GSWFs depicted in Figs. 1(a) and 1(b). The requirement that each electron sees the same number of flux quanta through the system equal to the number of electrons (we are at $\nu=1$) very much reduces the number of possible states—wave functions in the mixed CB-CF approach. We can consider, for example, the possibility (a) depicted in Fig. 1 which stands for the following wave function in the LLL:

$$\begin{aligned}\Psi_1=\mathcal{P}\mathcal{A}_\uparrow\mathcal{A}_\downarrow\Bigg\{&\prod_{i<j}(z_{i\uparrow}-z_{j\uparrow})\prod_{k<l}(z_{k\downarrow}-z_{l\downarrow})\prod_{p,q}(z_{p\uparrow}-z_{q\downarrow})\\&\times\mathcal{F}_s(w_\uparrow,\bar{w}_\uparrow)\prod_{i<j}(w_{i\uparrow}-w_{j\uparrow})^2\\&\times\mathcal{F}_s(w_\downarrow,\bar{w}_\downarrow)\prod_{k<l}(w_{k\downarrow}-w_{l\downarrow})^2\\&\times\prod_{i,j}(z_{i\uparrow}-w_{j\uparrow})\prod_{k,l}(z_{k\uparrow}-w_{l\downarrow})\\&\times\prod_{p,q}(z_{p,\downarrow}-w_{q,\uparrow})\prod_{m,n}(z_{m\downarrow}-w_{n\downarrow})\Bigg\},\end{aligned} \tag{3}$$

where $\mathcal{A}_\uparrow$ and $\mathcal{A}_\downarrow$ denote the overall antisymmetrizations. In the thermodynamic limit, the relation between the number of particles and flux quanta reads

$$N_\phi^b = N_{b\uparrow} + N_{b\downarrow} + N_{f\uparrow} + N_{f\downarrow},$$

$$N_\phi^{f\uparrow} = 2N_{f\uparrow} + N_{b\uparrow} + N_{b\downarrow},$$

$$N_\phi^{f\downarrow} = 2N_{f\downarrow} + N_{b\uparrow} + N_{b\downarrow}, \tag{4}$$

where we denoted by N_ϕ^b and $N_\phi^{f\sigma}$ separately the number of flux quanta that electrons that correlate as CBs and CFs see, respectively, and $N_{b\sigma}$ and $N_{f\sigma}$ are the number of CBs and CFs inside the layer σ, respectively ($\sigma = \uparrow, \downarrow$ is the layer index). The requirement constrains $N_\phi = N_\phi^b = N_\phi^{f\sigma}$, where N_ϕ is the number of flux quanta through the system. (This leads to the additional requirement $N_{f\uparrow} = N_{f\downarrow}$ which leaves $N_{b\uparrow} - N_{b\downarrow}$ unconstrained, connected with the Bose condensation phenomenon that the wave function should be part of.[4,13,18])

The only additional way to count the flux quanta that electrons see, with the (symmetric under $\uparrow \leftrightarrow \downarrow$ reversal) application of the Jastrow-Laughlin factors that we need to have, is

$$N_\phi^{b\uparrow} = N_{b\uparrow} + N_{b\downarrow} + 2N_{f\uparrow},$$

$$N_\phi^{b\downarrow} = N_{b\uparrow} + N_{b\downarrow} + 2N_{f\downarrow},$$

$$N_\phi^{f\uparrow} = 2N_{f\uparrow} + 2N_{b\uparrow},$$

$$N_\phi^{f\downarrow} = 2N_{f\downarrow} + 2N_{b\downarrow}, \tag{5}$$

which leads to the possibility (b) (with constraints $N_{b\uparrow} = N_{b\downarrow}$ and $N_{f\uparrow} = N_{f\downarrow}$). The intercorrelations in the first family in Fig. 1(a) are in the spirit of Ψ_{111} correlations, and those in the second family in Fig. 1(b) are in the spirit of the decoupled state, $\Psi_{1/2} \times \Psi_{1/2}$, where we correlate exclusively inside each layer.

C. Discussion

We can imagine a mixture of both intercorrelations [of Fig. 1(a) and Fig. 1(b)] in a single wave function but these mixed states, by their basic response,[18] fall into one of the universality classes depicted in Fig. 1. In Ref. 18 explicitly such a mixture and possibility under name "generalized vortex metal" was considered, in the scope of a Chern-Simons (CS) theory, and it was proved that it does not support a Goldstone (gapless) mode which was found to exist for the state depicted in Fig. 1(a). These generalized states belong to the universality class of the state depicted in Fig. 1(b) for which in the scope of the same theory we find in the low-energy spectrum only a gapped collective mode.[18]

The Chern-Simons theory we mentioned neglects the overall antisymmetrization built in the classes of Fig. 1. We can justify this neglect (1) by taking a point of view that stems from similar situations with quantum Hall states like hierarchy and Jain's constructions that in the low-energy sector can be considered as multicomponent systems[19] [we will argue later that the state of Eq. (3) can be mapped to a hierarchy construction], or (2) *a posteriori* because the results of the effective description of the classes in Fig. 1 are quite sensible and are expected for the states we are familiar with from numerics [the state in our Fig. 1(a) as analyzed in Ref. 16]. (We do not ask this type of theory for detailed answers anyway.) In this way it was found by us (Refs. 18 and 20), examining the basic response in the pseudospin channel in the random-phase approximation (RPA) of these Chern-Simons theories that the states in Figs. 1(a) and 1(c) represent superfluids, and the states in Figs. 1(b) and 1(d) represent disordered superfluids, compressible and incompressible, respectively. [Later, in a more complete study, we will find that the states of Fig. 1(d) are also compressible in the neutral channel.]

The two basic possibilities of connecting two extremes as depicted in Fig. 1, i.e., without and with pairing of CFs, must correspond to the two possible ways or paradigms that we know of disordering a superfluid. We will substantiate this claim further by examining the two superfluid constructions [Figs. 1(a) and 1(c)] in more detail.

III. NEUTRAL FERMIONS AND BKT DISORDERING

A. Dual picture of the first family of wave functions with neutral fermions

Let us write out the unprojected in the LLL version of the construction in Fig. 1(a) [Eq. (3)] in the following way:

$$\Psi_1 = \mathcal{A}_\uparrow \mathcal{A}_\downarrow \Bigg\{ \Psi_{111}(z_\uparrow, z_\downarrow)\Psi_{1/2}(w_\uparrow)\Psi_{1/2}(w_\downarrow) \times \prod_{i,j}(z_{i\uparrow} - w_{j\uparrow}) \prod_{k,l}(z_{k\uparrow} - w_{l\downarrow}) \times \prod_{p,q}(z_{i\downarrow} - w_{q\uparrow}) \prod_{m,n}(z_{m\downarrow} - w_{n\downarrow}) \Bigg\}, \tag{6}$$

where, as before, z_σ's and w_σ's denote coordinates of electrons belonging to the layer with index σ and $\mathcal{A}_\uparrow$ and $\mathcal{A}_\downarrow$, as before, stand for the antisymmetrizations. Using $\mathcal{S}_\uparrow$ and $\mathcal{S}_\downarrow$ symmetrizers inside each layer, the same function, Ψ_1, can be written as

$$\Psi_1 = \mathcal{S}_\uparrow \mathcal{S}_\downarrow \Bigg\{ \frac{\prod_{k<l}(w_{k\uparrow} - w_{l\uparrow}) \prod_{p<q}(w_{p\downarrow} - w_{q\downarrow})}{\prod_{i,j}(w_{i\uparrow} - w_{j\downarrow})} \times \mathcal{F}_s(w_\uparrow)\mathcal{F}_s(w_\downarrow) \Bigg\} \Psi_{111}, \tag{7}$$

where Ψ_{111} denotes the Vandermonde determinant (Slater determinant in the LLL) of all coordinates in which all groups equally participate.

By using the expressions for the densities of electrons in each layer, $\rho^\sigma(\eta) = \Sigma_i \delta^2(\eta - z_i^\sigma)$, *here now* z_σ's denote all electrons of the layer σ, we can rewrite the wave function in the following way:

$$\Psi_1 = \int d^2\eta_{1\uparrow}\cdots\int d^2\eta_{n\downarrow} \times \frac{\prod_{k<l}(\eta_{k\uparrow}-\eta_{l\uparrow})\prod_{p<q}(\eta_{p\downarrow}-\eta_{q\downarrow})}{\prod_{i,j}(\eta_{i\uparrow}-\eta_{j\downarrow})}\mathcal{F}_s(\eta_\uparrow) \times\mathcal{F}_s(\eta_\downarrow)\rho^\uparrow(\eta_{1\uparrow})\cdots\rho^\downarrow(\eta_{n\downarrow})\Psi_{111}(z_\uparrow,z_\downarrow), \tag{8}$$

where n is the total number of electrons that correlate as CFs. The equality is exact; any time we have in the product of ρ's the same layer electron coordinate more than once, the Laughlin-Jastrow factors of η's in the same layer force the wave function to become zero. The expression in Eq. (8) reminds us of a dual picture in terms of some quasiparticles with η coordinates as in Ref. 21. Certainly we are not describing the incompressible physics of a Laughlin state where quasihole operators of coherent states span the basis of low-energy physics and allow the description in terms of wave functions of quasiholes (dual picture).[21] Nevertheless we will argue that we can delineate a sector (find a subspace) to which constructions [Eq. (8) where n is arbitrary] belong, which is spanned by a quasiparticle basis of some neutral fermionic quasiparticles.

To find those quasiparticles we will rewrite Eq. (8) as

$$\Psi_1 = \int d^2\eta_{1\uparrow}\cdots\int d^2\eta_{n\downarrow} \times \frac{\prod_{k<l}|\eta_{k\uparrow}-\eta_{l\uparrow}|\prod_{p<q}|\eta_{p\downarrow}-\eta_{q\downarrow}|}{\prod_{i,j}|\eta_{i\uparrow}-\eta_{j\downarrow}|}\mathcal{F}_s(\eta_\uparrow)\mathcal{F}_s(\eta_\downarrow) \times\{\exp\{i\phi(\eta_{1\uparrow}\cdots\eta_{n\downarrow})\}\rho^\uparrow(\eta_{1\uparrow})\cdots\rho^\downarrow(\eta_{n\downarrow})\Psi_{111}(z_\uparrow,z_\downarrow)\}, \tag{9}$$

where $\exp\{i\phi(\eta)\}$ factor denotes the phase part of the Laughlin-Jastrow factors in front of the Fermi seas in Eq. (8). With respect to Eq. (8) we are allowed to take for definiteness that the phase factor always vanishes when any of two η's (or more) from the same layer coincide.

Our first question may be why states as

$$\rho^\uparrow(\eta_{1\uparrow})\cdots\rho^\downarrow(\eta_{n\downarrow})\Psi_{111}\sim\Psi_b(\eta_{1\uparrow},\ldots,\eta_{n\downarrow}) \tag{10}$$

would not make a bosonic basis. We look for the following overlap:

$$\int dz_{1\uparrow}\cdots\int dz_{N\downarrow}\Psi_b(\eta'_{1\uparrow},\ldots,\eta'_{n\downarrow})\Psi_b(\eta_{1\uparrow},\ldots,\eta_{n\downarrow}). \tag{11}$$

In the expansion of the density sums we may get

$$\delta^2(\eta'_1-z_1^\uparrow)\delta^2(\eta'_2-z_1^\uparrow)\delta^2(\eta_1-z_1^\uparrow)\delta^2(\eta_2-z_2^\uparrow)\cdots, \tag{12}$$

which would lead to the following contribution after z integration:

$$\delta^2(\eta'_1-\eta'_2)\delta^2(\eta_1-\eta'_1)|\eta_1-\eta_2|^2\exp\left\{-\frac{1}{2}(|\eta_1|^2+|\eta_2|^2)\right\} \times\frac{1}{|\eta_1-\eta_2|^2}\exp\left\{\frac{1}{2}(|\eta_1|^2+|\eta_2|^2)\right\}\cdots. \tag{13}$$

The last term, before the dots, comes after the integration over z's that do not participate in the delta functions. As usual[21] the term is the result of the screening of plasma which we find in the plasma analogy of Ψ_{111} state in its charge channel. The term exactly cancels the preceding one (it is equal to its inverse) and the same cancellation will happen for any pair of η's (in the place of …) that in remaining z integration have the role of impurities (of charge one) in the plasma of remaining z's. This is very good because of our goal to find basis states and leaves us to consider only delta functions in the contribution. But we can see immediately in Eq. (13) that $\delta(\eta'_1-\eta'_2)$ spoils our goal that the states mimic a Fock basis of bosonic particles. Therefore as candidates for basis states we should consider fermionic states,

$$|\eta_{1\uparrow}\cdots\eta_{n\downarrow}\rangle = \frac{1}{\sqrt{n!\binom{N}{n}}}\exp\{i\phi(\eta_{1\uparrow}\cdots\eta_{n\downarrow})\} \times\rho^\uparrow(\eta_{1\uparrow})\cdots\rho^\downarrow(\eta_{n\downarrow})|\Psi_{111}\rangle \tag{14}$$

for which we cannot get contributions of the type in Eq. (13) because the phase part does not allow two (or more) quasiparticles to coincide. [Eq. (14) represents a fermionic state for η quasiparticles because of the phase part introduced in Eq. (9) which is antisymmetric under the exchange of η's.] Therefore we should consider fermionic states in Eq. (14) because of the previously found nondesirable terms in the bosonic case (we are looking for quasiparticles and their basis states that would have features of the Fock space basis): the terms like the one with $\delta(\eta'_1-\eta'_2)$ in Eq. (13) lead to the absence of orthogonality of these states, which we would like to represent coordinate basis states in the bosonic case and that can be mended by taking fermions—then these terms are absent. By a similar analysis which lead to Eq. (13), considering various possibilities for delta function contributions of density operators we can find that the leading most singular and coherent behavior of the states defined in Eq. (14) is

$$\langle\eta'_{1\uparrow},\eta'_{2\uparrow}\cdots\eta'_{n\downarrow}|\eta_{1\uparrow},\eta_{2\uparrow}\cdots\eta_{n\downarrow}\rangle\to\delta^2(\eta'_{1\uparrow}-\eta_{1\uparrow}) \times\delta^2(\eta'_{2\uparrow}-\eta_{2\uparrow})\cdots\delta^2(\eta'_{n\downarrow}-\eta_{n\downarrow})-\delta^2(\eta'_{1\uparrow}-\eta_{2\uparrow}) \times\delta^2(\eta'_{2\uparrow}-\eta_{1\uparrow})\cdots\delta^2(\eta'_{n\downarrow}-\eta_{n\downarrow})+\cdots. \tag{15}$$

The rest of contribution constitute incoherent phase factors with fewer number ($<n$) of delta functions but of the same kind as in the leading behavior. We cannot prove that the states make exactly a Fock space of neutral fermionic quasiparticles, i.e., we do not have an exact equality in Eq. (15), but they stand fairly close to that status. In other words we do not have the exact equality in Eq. (15), i.e., equality to delta functions only (appropriately antisymmetrized), but we have in addition some finite contributions which cannot change the fact that the overlap is singular—at its maximum

when η''s and η's coincide. Therefore quasiparticles are not pointlike fermionic quasiparticles (one certainly cannot expect that from quasiparticles in a strongly correlated system); they are extended, but clearly the overlap has the singular contribution of antisymmetrized delta functions which points out that we are fairly (to a good extent) close to the fermionic Fock basis description. Even in the Laughlin case we cannot prove the exact LLL delta function overlaps of coherent states of quasiholes. The quasiparticles are neutral because in the construction of the states there is no net magnetic flux through the system. See Eq. (14) and the definition of the phase factor in Eq. (9) with Eq. (8).

Now that we know basis states just by looking at Eq. (9) we can read out the GSWF in the dual picture in terms of neutral fermions,

$$\Psi_{\text{dual}}(\eta) = \frac{\prod_{k<l}|\eta_{k\uparrow}-\eta_{l\uparrow}|\prod_{p<q}|\eta_{p\downarrow}-\eta_{q\downarrow}|}{\prod_{i,j}|\eta_{i\uparrow}-\eta_{j\downarrow}|}\mathcal{F}_s(\eta_\uparrow)\mathcal{F}_s(\eta_\downarrow). \tag{16}$$

This is a wave function of a 2D Coulomb fermionic plasma. In the literature 2D Coulomb fermionic plasma with same charge particles is fairly known and explored.[22,23] It is a dynamical system of fermionic particles in 2D that interact with the long-range ($\sim -\ln\{r\}$) interaction. As shown in Ref. 22 the Jastrow factor of the type $\prod_{i<j}|z_i - z_j|^\gamma$ (γ proportional to the interaction coupling constant), together with multiplying Slater determinant of free waves, describes the ground-state function in the long-distance limit. In our case we have a generalization of such a system to the one with opposite charges. Assuming that the concentration of particles is not large, which is the case of interest to us, we then expect the dipole configurations of particles that the wave function in Eq. (16) describes.

B. Discussion

Merons are true elementary vorticity quasiparticles of the translatory invariant QHB system at least for small distances between layers as shown in Ref. 4 and carry both charge and vorticity. Therefore the neutral fermion basis that we described can be a complete basis for the ground-state evolution of the QHB in the nontranslatory invariant case in which merons by their charges are bound to impurities.

The wave function in Eq. (16) describes the superfluid state in Fig. 1(a). It encodes dipole positioning of opposite vorticity (layer index) neutral fermions. With increasing distance there are more dipoles of neutral fermions and they are expected to be less tightly bound as in the description of a BKT disordering of a 2D system with increasing temperature. Therefore we do not find quantum fluctuations in this case. This will be explicitly shown by calculations in the following section (see also Appendix A).

In the superfluid phase, with respect to merons, a neutral fermion dipole should be in essence a superposition of quadrupolar combinations of merons—two dipoles which come in pairs but at arbitrary distance as illustrated in Fig. 2. In this way, as special configurations of dipoles, neutral fermions, we expect, constitute the lowest lying states of the QHB—(pseudo)spin or phonon waves.[2,14]

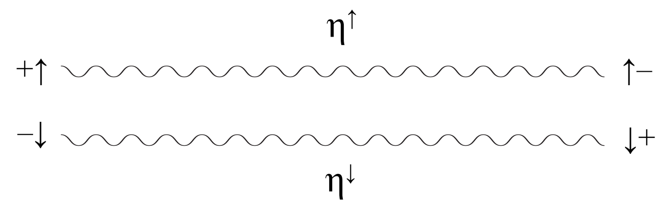

FIG. 2. The quadrupolar configurations of merons that make neutral fermion pair. Compare the same configuration of Laughlin quasiparticles as a description of "magnetophonon" branch in the Laughlin case in Ref. 14.

If neutral fermions may be considered as eigenstates they must lie very high in spectrum; like electrons in fractional quantum Hall states they constitute the physics of Ψ_1 but their wave function Eq. (16) describes a highly correlated state.

The dual expression of Eq. (16) was derived under assumption of the screening properties in the charge channel of the particles participating in the plasma analogy based on Ψ_{111} state. As the distance is increased there are less of them and the breakdown of the description in terms of dipoles of neutral fermions at smaller distances becomes a possibility. We expect that due to impurities there will be patches (islands) of dissociated neutral fermions.[24]

IV. QUANTUM FLUCTUATIONS AND QUANTUM DISORDERING

A. Introduction

The two paradigms-models of superfluid disordering as applied to our 2+1 dimensional system mean that the time evolution is such that (1) meron-antimeron pairs are locked on impurities or (2) created and annihilated at some later time and therefore making a loop in time. The loops in time signify the presence of quantum disordering.[7] We will discuss and detect the presence of quantum disordering in the WFs of class (*c*) in Fig. 1 by examining how they relate to and incorporate ordinary (not quantum disordering that involves merons-vortices) quantum fluctuation phonon contribution in this case.[6,8] We will find that the WFs of class (*a*) in Fig. 1 do not have this contribution.

B. Quantum fluctuations due to phonons and quantum disordering

The usual[14] CS field theory approach[8] in the RPA to the bilayer problem at $\nu=1$ (which in the neutral channel reduces just to the problem of ordinary superfluid with only phonon description and contribution) finds the following correction to the Ψ_{111} state:

$$\Psi_{\text{PH}} = \exp\left\{-\frac{1}{2}\sum_k \frac{\sqrt{\frac{V_-(k)}{\rho_E}}}{k}\rho_k^- \rho_{-k}^-\right\}\Psi_{111}, \tag{17}$$

where $\rho_k^- = \rho_k^\uparrow - \rho_k^\downarrow$, $V_-(k) = \frac{V_{\uparrow\uparrow}(k) - V_{\uparrow\downarrow}(k)}{2}$, $V_{\uparrow\uparrow} = \frac{2\pi}{k}$, $V_{\uparrow\downarrow} = \frac{2\pi}{k}\exp\{-kd\}$, i.e., $V_-(k)$ is the interaction in the neutral channel, ρ_E

$=\frac{\bar{\rho}}{m}$, where m is the electron mass and $\bar{\rho}$ is the uniform total density. In the small d limit $V_{-}(k)=\pi d$ and we can expand the expression $\Psi_{\rm PH}$ as

$$\Psi_{\rm PH}=\Psi_{111}-\left(\sum_{k}\frac{c\sqrt{d}}{k}\rho_{-k}^{-}\rho_{k}^{-}\right)\Psi_{111}+\cdots, \qquad (18)$$

where c is a positive constant. The terms after the first one represent corrections, in the order of importance, to the Ψ_{111} ansatz as d increases.

On the other hand the WFs of Fig. 1(c) are more general as they suggest the form of the correction terms of wider class than the one used in the expansion [Eq. (18)] with only exception that the class demands equal number of $\rho_k^{\uparrow}$'s and $\rho_k^{\downarrow}$'s because in writing down the classes of Fig. 1 we explicitly distinguished $\uparrow$'s from $\downarrow$'s and fixed the number of $\uparrow$'s and $\downarrow$'s.

We can start comparing and relating the first phonon correction, i.e.,

$$\sim\sum_{k}\frac{1}{k}\rho_{-k}^{\uparrow}\rho_{k}^{\downarrow} \qquad (19)$$

to a wave function of two neutral fermions ($\uparrow$ from $\downarrow$), i.e., density operators as in Eq. (8) but with a pairing between them as in class Fig. 1(c).

Without pairing we would have

$$\int d^2\eta_{1\uparrow}\int d^2\eta_{2\downarrow}\frac{1}{(\eta_{1\uparrow}-\eta_{2\downarrow})}\rho^{\uparrow}(\eta_{1\uparrow})\rho^{\downarrow}(\eta_{2\downarrow}), \qquad (20)$$

which is identical to zero (no correction) as can be found out in Appendix A. This is an important result and shows that there are no quantum fluctuations in the first family of WFs discussed in Sec. III. Besides this analytical proof, our statement is further corroborated by the fact that the computer-generated two neutral fermion state also does not exist—see Ref. 16.

Therefore we continue by considering

$$\int d^2\eta_{1\uparrow}\int d^2\eta_{2\downarrow}\frac{1}{|\eta_{1\uparrow}-\eta_{2\downarrow}|^{\alpha}}\rho^{\uparrow}(\eta_{1\uparrow})\rho^{\downarrow}(\eta_{2\downarrow}), \qquad (21)$$

where $\alpha=1$ if we take $g(z)=\sqrt{\frac{z}{z^*}}$ for the pairing function or $\alpha=2$ if $g(z)=\frac{1}{z^*}$ For $\alpha=1$ the expression in Eq. (20) reduces to the form of the first phonon contribution in the long-distance limit with the $\frac{1}{k}$ singularity (see Appendix A) and for $\alpha=2$ this singularity softens to $\sim-\ln\{kl_B\}$ where l_B is the magnetic length (see Appendix A). We will consider only these most weakly pairing cases; the case $g(z)=\frac{1}{z}$ does not produce correction as can be seen in Appendix A.

Next we consider more than two density operator constructions, i.e., more than two neutral fermions constructions as in Eq. (8) but instead of the two decoupled Fermi seas we have a pairing between neutral fermions,

$$\begin{aligned}\Psi_2^n=&\int d^2\eta_{1\uparrow}\cdots\int d^2\eta_{n\downarrow}\\ &\times\frac{\prod_{k<l}(\eta_{k\uparrow}-\eta_{l\uparrow})\prod_{p<q}(\eta_{p\downarrow}-\eta_{q\downarrow})}{\prod_{i,j}(\eta_{i\uparrow}-\eta_{j\downarrow})}\\ &\times\mathrm{Det}\{g(\eta_{\uparrow}-\eta_{\downarrow})\}\rho^{\uparrow}(\eta_{1\uparrow})\cdots\rho^{\downarrow}(\eta_{n\downarrow})\Psi_{111}(z_{\uparrow},z_{\downarrow}),\\ =&\int d^2\eta_{1\uparrow}\cdots\int d^2\eta_{n\downarrow}\mathrm{Det}\left\{\frac{1}{\eta_{\uparrow}-\eta_{\downarrow}}\right\}\\ &\times\mathrm{Det}\left\{\sqrt{\frac{\eta_{\uparrow}-\eta_{\downarrow}}{\eta_{\uparrow}^*-\eta_{\downarrow}^*}}\right\}\times\rho^{\uparrow}(\eta_{1\uparrow})\cdots\rho^{\downarrow}(\eta_{n\downarrow})\Psi_{111}(z_{\uparrow},z_{\downarrow}),\end{aligned} \qquad (22)$$

where in the second expression we used the Cauchy determinant identity, i.e.,

$$\frac{\prod_{k<l}(\eta_{k\uparrow}-\eta_{l\uparrow})\prod_{p<q}(\eta_{p\downarrow}-\eta_{q\downarrow})}{\prod_{i,j}(\eta_{i\uparrow}-\eta_{j\downarrow})}=\mathrm{Det}\left\{\frac{1}{\eta_{\uparrow}-\eta_{\downarrow}}\right\} \qquad (23)$$

and substituted the pairing function that has lead us to the first phonon correction for two paired neutral fermions. Immediately we can see that the diagonal terms in which pairs of the two determinants are the same would make further phononlike corrections, i.e., their superposition with appropriate coefficients would lead to

$$\exp\left\{-\sum_{k}\frac{c_d}{k}\rho_{-k}^{\uparrow}\rho_{k}^{\downarrow}\right\}\Psi_{111}. \qquad (24)$$

The other nondiagonal terms would lead to more complicated constructions of four and more neutral fermions that should participate in the description of quantum disordering, i.e., describe the physics beyond phonon contribution (24). Although in some sense we are talking just about a class (a pool) of wave functions that should describe quantum disordering we can fix general form, at least for small d, of the superposition that should completely model the ground state at fixed d

$$\Psi_0=\sum_{n=0,2,\ldots}\Psi_2^n c_n. \qquad (25)$$

In the long-distance limit (25) should tend to Eq. (24). In other words nondiagonal terms in Eq. (22) should be subleading to the leading behavior in Eq. (24). That this is true from the physical point of view we expect that it is enough to prove the subleading behavior in the case of four neutral fermions (Ψ_2^4) and that can be found in Appendix B. The proof is based on the smallness of higher-order terms that may appear inside the brackets in Eq. (24). This is assumed in the RPA approach and expected in the small d limit.

Therefore the quantum Hall physics besides $g(z)=\sqrt{\frac{z}{z^*}}$ pairing possibility brings or allows the possibility of $g(z)=\frac{1}{z^*}$ pairing that introduces nontrivial quantum corrections, i.e., brings another kind of quantum disordering. The $g(z)$

$=\sqrt{\frac{z}{z^*}}$ accommodates the usual (on the level of RPA) superfluid description in which we may expect that the disordered phase will break translation symmetry. Indeed, the bosonic CS field theories that are not based on quantum Hall WFs give this scenario of the disordered phase as a charge-density wave.[8] It seems, therefore, there are two possible scenarios for superfluid disordering not in the BKT class for the bilayer in the translation symmetry invariant case (without impurities). In the following we will discuss the second possibility with $g(z)=\frac{1}{z^*}$ kind of pairing.

C. Weak pairing $g(z)\sim\frac{1}{z^*}$ case and conformal field theory considerations

We expect, if the translational symmetry of the ground state remains unbroken, that also in the case of pairing $g(z)=\frac{1}{z^*}$ the translatory invariant system smoothly evolves with the increase in d into the class of wave functions in Fig. 1(d). We would like to know more about this class—whether it represents a distinct phase. If we take the choice $g(z)=\frac{1}{z^*}$ and examine the final form of the state of Fig. 1(d) when there are no CBs, we are lead to its following forms:

$$\begin{aligned}\Psi_2 &= \mathrm{Det}\left\{\frac{1}{z_{i\uparrow}^*-z_{j\downarrow}^*}\right\}\prod_{i<j}(z_{i\uparrow}-z_{j\uparrow})^2\prod_{k<l}(z_{k\downarrow}-z_{l\downarrow})^2\\ &= \mathrm{Det}\left\{\frac{1}{z_{i\uparrow}^*-z_{j\downarrow}^*}\right\}\mathrm{Det}\left\{\frac{1}{z_{k\uparrow}-z_{l\downarrow}}\right\}\Psi_{111},\end{aligned} \tag{26}$$

where to get the last line we used the Cauchy determinant identity. The neutral part of Ψ_2 (not carrying a net flux through the system as Ψ_{111} does) that consists of the two determinants can be viewed as a correlator of vertex operators of a single *nonchiral* bosonic field. According to[25] conformal field theory (CFT) correlators not only describe quantum Hall system WFs but also can be used to find out about excitation spectrum and connect to its edge and bulk theories. In this way motivated neutral excitations are vertex operators that correspond to single-valued WF expressions that multiply Ψ_2,

$$\exp\{i\delta_1\phi(w,w^*)\}\rightarrow\frac{\prod_i|z_{i\uparrow}-w|^{2\delta_1}}{\prod_i|z_{i\downarrow}-w|^{2\delta_1}}, \tag{27}$$

$$\exp\{i\delta_2\theta(w,w^*)\}\rightarrow\frac{\prod_i(z_{i\uparrow}-w)^{\delta_2}}{\prod_i(z_{i\uparrow}^*-w^*)^{\delta_2}}\frac{\prod_i(z_{i\downarrow}^*-w^*)^{\delta_2}}{\prod_i(z_{i\downarrow}-w)^{\delta_2}}, \tag{28}$$

where $\phi(w,w^*)=\phi(w)+\phi(w^*)$, $\theta(w,w^*)=\phi(w)-\phi(w^*)$, and $\phi(w)$ and $\phi(w^*)$ are holomorphic and antiholomorfic parts of the bosonic field, respectively. δ_2 must be $\frac{1}{2}$ because of the requirement of single-valuedness. For detailed explanations of the bosonic CFT analogies see Appendix C.

If the low-lying spectrum were consisting only of $\delta_1=\frac{1}{2}$ and $\delta_2=\frac{1}{2}$ quasiparticle excitations our system would be described by so-called BF Chern-Simons theory or the theory of 2D superconductor.[26] The mutual statistics of quasiparticles-quasiparticles and vortices in this theory is semionic (due to the fact that vortices carry half-flux $(\frac{h}{(2e)c})$ quantum) and that this is also the case with our excitations can be easily checked via CFT correlators—see Appendix C. Combining the analysis with the charge part (Ψ_{111}) in which only charge 1 excitations are allowed (half-flux quantum excitations are strongly confined[27]) we may come to the conclusion that the degeneracy of the system GSs on the torus must be 4 (Refs. 26 and 28) But the expression for the first kind of excitations [Eq. (27)] allows a real continuum for the value of δ_1 exponent including $\delta_1=0$, and therefore we expect a compressible (gapless) behavior of the system despite the incompressibility of the charge channel and seemingly topological phase behavior in the neutral sector. Nevertheless we expect that in our case BF CS theory is a part of the description of the pairing phase in a Lagrangian in which there is a quadratic nonderivative term in one of the two gauge fields; this allows a branch of gapless excitations—see Appendix C for details.

The question may come why we did not do an analysis with the projection to the LLL. Certainly the analysis is more involved where "reversed flux part," i.e., complex conjugated determinant becomes an operator that acts on the rest of wave function. Nevertheless, an analysis of the edge excitation spectrum[29] suggests that it cannot conform to any description of simple free CFT theories, i.e., cannot belong to a totally incompressible class, and it is very likely that the system is, as it follows from our unprojected analysis, compressible in the neutral channel. Therefore it is very hard to distinguish the physics of the states in Figs. 1(c) and 1(d) in the translatory invariant system that involve pairing of the type $g(z)=\frac{1}{z^*}$.

While we were finishing the writing a numerical study (of homogenous WFs in the translatory invariant case) appeared[30] that agrees with and complements our conclusions on the nature of pairing.

V. CONCLUSIONS

In conclusion, we presented two families of wave functions that describe two possible ways of homogeneous disordering of the quantum Hall superfluid with their detailed description on the basis of the dual (quasiparticle) picture of the quantum Hall effect. We also presented detailed analysis of the disordering in the translation invariant system on the basis of insights into the pairing function of quasiparticles-neutral fermions. A class of candidate wave functions was clearly connected with the formalism that we find in other (Chern-Simons) theories, and the pairing function $g(z)\sim\frac{1}{z^*}$ was extracted as a clear choice that incorporates quantum disordering and that will describe the system if it does not transform into a CDW (charge-density wave) inhomogeneous solution.

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APPENDIX A

We want to prove

$$\int d^2\eta_{1\uparrow}\int d^2\eta_{2\downarrow}\frac{1}{(\eta_{1\uparrow}-\eta_{2\downarrow})}\rho^{\uparrow}(\eta_{1\uparrow})\rho^{\downarrow}(\eta_{2\downarrow})=0. \quad \text{(A1)}$$

After switching to the Fourier space, $\rho^{\sigma}(\eta)=\Sigma_k\rho_k^{\sigma}\exp\{i\vec{k}\vec{\eta}\}$, the left-hand side (l.h.s.) becomes

$$2\pi\sum_k\int d^2\eta\frac{1}{\eta}\exp\{i\vec{k}\vec{\eta}\}\rho_k^{\uparrow}\rho_{-k}^{\downarrow}. \quad \text{(A2)}$$

The angle part of the integration with the help of the table integral[31]

$$\int_0^{\pi}\exp\{i\beta\cos x\}\cos\{nx\}=i^n\pi J_n(\beta) \quad \text{(A3)}$$

yields

$$\begin{aligned}\int d^2\eta\frac{1}{\eta}\exp\{i\vec{k}\vec{\eta}\}&=\int_0^{\infty}dr[i\pi J_1(kr)-i\pi J_1(-kr)]\\&=(-)\frac{i\pi}{k}\int_0^{\infty}dr\left[\frac{dJ_0(kr)}{dr}+\frac{dJ_0(-kr)}{dr}\right]\\&=i\frac{2\pi}{k},\end{aligned} \quad \text{(A4)}$$

where we used notation $|\vec{k}|=k$ and in the last line the identity for the Bessel functions: $J_0'=-J_1$. On the other hand

$$\int d^2\eta\frac{1}{\eta}\exp\{-i\vec{k}\vec{\eta}\}=-i\frac{2\pi}{k}, \quad \text{(A5)}$$

and therefore Eq. (A2) can be written as

$$\begin{aligned}&\pi\sum_k\left[\int d^2\eta\frac{1}{\eta}\exp\{i\vec{k}\vec{\eta}\}\rho_k^{\uparrow}\rho_{-k}^{\downarrow}+\int d^2\eta\frac{1}{\eta}\exp\{-i\vec{k}\vec{\eta}\}\rho_{-k}^{\uparrow}\rho_k^{\downarrow}\right.\\&=i2\pi^2\sum_k\frac{1}{|\vec{k}|}(\rho_k^{\uparrow}\rho_{-k}^{\downarrow}-\rho_{-k}^{\uparrow}\rho_k^{\downarrow})=0\quad\text{QED}.\end{aligned} \quad \text{(A6)}$$

Next we want to evaluate

$$\int d^2\eta_{1\uparrow}\int d^2\eta_{2\downarrow}\frac{1}{|\eta_{1\uparrow}-\eta_{2\downarrow}|^{\alpha}}\rho^{\uparrow}(\eta_{1\uparrow})\rho^{\downarrow}(\eta_{2\downarrow}). \quad \text{(A7)}$$

Again this reduces in the Fourier space to

$$2\pi\sum_k\int d^2\eta\frac{1}{|\eta|^{\alpha}}\exp\{i\vec{k}\vec{\eta}\}\rho_k^{\uparrow}\rho_{-k}^{\downarrow}. \quad \text{(A8)}$$

In the case of $\alpha=1$ as usual for the real Coulomb interaction in 2D the integral is

$$\int d^2\eta\frac{1}{|\eta|}\exp\{i\vec{k}\vec{r}\}=2\pi\int_0^{\infty}drJ_0(kr)=\frac{2\pi}{k}. \quad \text{(A9)}$$

In the case of $\alpha=2$ we have

$$\int d^2\eta\frac{1}{|\eta|^2}\exp\{i\vec{k}\vec{r}\}=2\pi\int_0^{\infty}dr\frac{J_0(kr)}{r}. \quad \text{(A10)}$$

The integral needs a cutoff at small distances (otherwise diverges) which should be included in our effective description and as usual can be taken to be l_B (magnetic length distance). Therefore, instead of Eq. (A10) we have

$$2\pi\int_0^{\infty}dr\frac{rJ_0(kr)}{r^2+l_B^2}=2\pi K_0(l_Bk). \quad \text{(A11)}$$

In the small momentum limit we can approximate

$$K_0(z)\approx-\ln\left\{\frac{z}{2}\right\}+o(z) \quad \text{(A12)}$$

and therefore our first phononlike correction in this case of pairing is

$$\sum_k(-)\ln\{kl_B\}\rho_k^{\uparrow}\rho_{-k}^{\downarrow}. \quad \text{(A13)}$$

For the case of pairing $g(z)=\frac{1}{z}$ we have

$$\int d^2\eta_{1\uparrow}\int d^2\eta_{2\downarrow}\frac{1}{(\eta_{1\uparrow}-\eta_{2\downarrow})^2}\rho^{\uparrow}(\eta_{1\uparrow})\rho^{\downarrow}(\eta_{2\downarrow}), \quad \text{(A14)}$$

which reduces to the solving of the following Fourier transform

$$\int d^2\eta\frac{1}{\eta^2}\exp\{i\vec{k}\vec{\eta}\}. \quad \text{(A15)}$$

With the help of Eq. (A3) we have for the value of the integral

$$\int_0^{\infty}dr\frac{1}{r}\{-\pi J_2(kr)-\pi J_2(-kr)\}. \quad \text{(A16)}$$

We may use the table integral[31]

$$\int_0^{\infty}\frac{J_{\nu}(ax)}{x^{\nu-q}}dx=\frac{\Gamma\left(\frac{1}{2}q+\frac{1}{2}\right)}{2^{\nu-q}a^{q-\nu+1}\Gamma\left(\nu-\frac{1}{2}q+\frac{1}{2}\right)} \quad \text{(A17)}$$

for $-1<Req<Re\nu-\frac{1}{2}$ to find out that the value of the integral does not depend on k, i.e.,

$$\int d^2\eta\frac{1}{\eta^2}\exp\{i\vec{k}\vec{r}\}=-\pi. \quad \text{(A18)}$$

Therefore the phononlike correction in this case is proportional to

$$\left(\sum_k\rho_k^{\uparrow}\rho_k^{\downarrow}\right)\Psi_{111} \quad \text{(A19)}$$

and in the real (coordinate) space this becomes

$$\begin{aligned}\int d^2\eta\rho^{\uparrow}(\eta)\rho^{\downarrow}(\eta)\Psi_{111}&=\sum_{i,j}\int d^2\eta\delta^2(\eta-z_{i\uparrow})\delta^2(\eta-z_{j\downarrow})\Psi_{111}\\&=\sum_{i,j}\delta^2(z_{i\uparrow}-z_{j\downarrow})\Psi_{111}=0,\end{aligned} \quad \text{(A20)}$$

i.e., no correction at all.

APPENDIX B

We consider nondiagonal (nonphononlike) corrections that come from the description of quantum disordering by the class of WFs in Fig. 1(c) when the pairing is fixed to be $g(z)=\sqrt{\frac{z}{z^*}}$, i.e., nondiagonal terms of Eq. (22) with $n=4$. We want to prove the subleading behavior with respect to the diagonal terms as the one with $\eta_{1\uparrow}\equiv\eta_1$, $\eta_{3\uparrow}\equiv\eta_3$, $\eta_{2\downarrow}\equiv\eta_2$, and $\eta_{4\uparrow}\equiv\eta_4$ in

$$\int d^2\eta_1\int d^2\eta_3\int d^2\eta_2\int d^2\eta_4\frac{1}{|\eta_1-\eta_2|}\frac{1}{|\eta_3-\eta_4|}\times\rho^{\uparrow}(\eta_1)\rho^{\uparrow}(\eta_3)\rho^{\downarrow}(\eta_2)\rho^{\downarrow}(\eta_4)=\left[\sum_k\frac{(2\pi)^2}{k}\rho_k^{\uparrow}\rho_{-k}^{\downarrow}\right]^2 \tag{B1}$$

of the following nondiagonal term

$$\int d^2\eta_1\int d^2\eta_3\int d^2\eta_2\int d^2\eta_4\frac{1}{(\eta_1-\eta_2)}\frac{1}{(\eta_3-\eta_4)}\times\sqrt{\frac{(\eta_1-\eta_4)(\eta_3-\eta_2)}{(\eta_1^*-\eta_4^*)(\eta_3^*-\eta_2^*)}}\rho^{\uparrow}(\eta_1)\rho^{\uparrow}(\eta_3)\rho^{\downarrow}(\eta_2)\rho^{\downarrow}(\eta_4). \tag{B2}$$

The nondiagonal terms by their forms should describe different processes from the phonon contributions, i.e., from those as $(\rho_{k_1}^{\uparrow}\rho_{-k_1}^{\downarrow})\cdots(\rho_{k_{n/2}}^{\uparrow}\rho_{-k_{n/2}}^{\downarrow})$ for arbitrary k's. In the long-distance approximation we will argue that the nondiagonal term [Eq. (B2)] carry less importance that the phonon contribution with the same number of density operators.

Introducing $\eta=\eta_1-\eta_4$, $\tilde{\eta}=\eta_3-\eta_2$, $\eta_-=\eta_1-\eta_3$ and $\eta_+=\eta_1+\eta_3$ we can rewrite Eq. (B2) as

$$\sum_{k_1,k_3,\tilde{k},k}\int d^2\eta\int d^2\tilde{\eta}\int d^2\eta_-\int d^2\eta_+\frac{1}{(\eta_-+\tilde{\eta})}\frac{1}{(\eta-\eta_-)}\times\sqrt{\frac{\eta\tilde{\eta}}{\eta^*\tilde{\eta}^*}}\exp\{i\eta_1\vec{k}_1\}\exp\{i\eta_3\vec{k}_3\}\times\exp\left\{i\left(\frac{1}{2}\vec{\eta}_+-\frac{1}{2}\vec{\eta}_--\vec{\tilde{\eta}}\right)\vec{\tilde{k}}\right\}\times\exp\left\{i\left(\frac{1}{2}\vec{\eta}_++\frac{1}{2}\vec{\eta}_--\vec{\eta}\right)\vec{k}\right\}\rho_{k_1}^{\uparrow}\rho_{k_3}^{\uparrow}\rho_{\tilde{k}}^{\downarrow}\rho_k^{\downarrow}. \tag{B3}$$

The η_+ integration brings the constraint $\vec{k}+\vec{\tilde{k}}+\vec{k}_1+\vec{k}_3=0$. Then the remaining η_- integration gives the following contribution:

$$\int d^2\eta_-\frac{1}{(\eta_-+\tilde{\eta})}\frac{1}{(\eta-\eta_-)}\exp\left\{i\frac{\vec{\eta}_-}{2}(\vec{k}_1-\vec{k}_3-\vec{\tilde{k}}+\vec{k})\right\}=-i\frac{2\pi}{|\vec{k}+\vec{k}_3|}\frac{1}{\eta+\tilde{\eta}}[\exp\{i\vec{\tilde{\eta}}(\vec{k}_3+\vec{\tilde{k}})\}-\exp\{-i\vec{\eta}(\vec{k}_3+\vec{\tilde{k}})\}], \tag{B4}$$

where we used the constraint. Therefore the contribution is proportional to

$$\sum_{k_3,\tilde{k},k}\frac{1}{|\vec{k}+\vec{k}_3|}\int d^2\eta\int d^2\tilde{\eta}\frac{1}{(\eta+\tilde{\eta})}\sqrt{\frac{\eta\tilde{\eta}}{\eta^*\tilde{\eta}^*}}\times[\exp\{i\vec{\tilde{\eta}}(\vec{k}_3+\vec{\tilde{k}})\}-\exp\{-i\vec{\eta}(\vec{k}_3+\vec{\tilde{k}})\}]\times\exp\{-i\tilde{\eta}\tilde{k}\}\exp\{-i\eta k\}\rho_{-\tilde{k}-k-k_3}^{\uparrow}\rho_{k_3}^{\uparrow}\rho_{\tilde{k}}^{\downarrow}\rho_k^{\downarrow}. \tag{B5}$$

In the long-distance limit $|\vec{k}+\vec{k}_3|\to 0$ but that does not cancel the part of the 2D volume in the integration measure like in the phonon contribution (that would damp the contribution) but is canceled by the difference of the exponentials in the same limit in Eq. (B5). There is only one more factor, i.e., $\frac{1}{\eta+\tilde{\eta}}$ that can bring the momentum inverse contribution but this only enforces $k\approx\tilde{k}$, i.e., $(\rho_k^{\uparrow}\rho_{-k}^{\downarrow})^2$ without a significant coefficient. This will only give the next order contribution inside the brackets in Eq. (24) which for small d, and as usual in the RPA approach, we can neglect.

APPENDIX C

We will give a more general view of the CFT analogies of so-called[32] doubled CS theories to which BF CS theory belongs. In the work of Freedman *et al.*[32] BF CS theory was classified as the low-energy theory of the deconfined phase of $\mathbb{Z}_2$ gauge theory. There also $SU(2)_1\times\overline{SU(2)}_1$ doubled CS theory was considered. For the detailed description of these theories the reader should consult Refs. 26 and 32. Here we will, by writing down relevant CFT correlators, demonstrate the analogies between nonchiral-complete CFTs and these doubled CS theories.

First we will consider $SU(2)_1\times\overline{SU(2)}_1$ case. The possible wave function with coordinates of two species $z_{1\uparrow},\ldots,z_{N\downarrow}$, for which there are equal number of $\uparrow$'s and $\downarrow$'s: $N_\uparrow=N_\downarrow$ and $N_\uparrow+N_\downarrow=N$, is

$$\Psi=\frac{\prod_{k<l}|z_{k\uparrow}-z_{l\uparrow}|\prod_{p<q}|z_{p\downarrow}-z_{q\downarrow}|}{\prod_{i,j}|z_{i\uparrow}-z_{j\downarrow}|}=\frac{\prod_{k<l}\sqrt{z_{k\uparrow}-z_{l\uparrow}}\prod_{p<q}\sqrt{z_{p\downarrow}-z_{q\downarrow}}}{\prod_{i,j}\sqrt{z_{i\uparrow}-z_{j\downarrow}}}\times\frac{\prod_{k<l}\sqrt{z_{k\uparrow}^*-z_{l\uparrow}^*}\prod_{p<q}\sqrt{z_{p\downarrow}^*-z_{q\downarrow}^*}}{\prod_{i,j}\sqrt{z_{i\uparrow}^*-z_{j\downarrow}^*}}. \tag{C1}$$

We use the following correlator of vertex operators of a bosonic field ϕ:

$$\langle\exp\{i\beta\phi(z_1,z_1^*)\}\exp\{-i\beta\phi(z_2,z_2^*)\}\rangle=\frac{1}{|z_1-z_2|^{2\beta^2}}. \tag{C2}$$

If $\alpha=\frac{1}{\sqrt{2}}$ we can rewrite our wave function as

$$\Psi=\langle\exp\{i\alpha\phi(z_1,z_1^*)\}\exp\{i\alpha\phi(z_2,z_2^*)\}\ldots\exp\{-i\alpha\phi(z_N,z_N^*)\}\rangle, \tag{C3}$$

and define

$$\phi(z,z^*) = \phi(z) + \phi(z^*), \tag{C4}$$

$$\theta(z,z^*) = \phi(z) - \phi(z^*). \tag{C5}$$

Inserting a neutral pair (w_1 and w_2) of $\exp\{i\delta_1\phi(w,w^*)\}$ vertex operators or $\exp\{i\delta_2\phi(w,w^*)\}$ vertex operators we can conclude that these insertions correspond to multiplying the wave function Ψ [Eq. (C1)] by

$$\exp\{i\delta_1\phi(w,w^*)\} \to \frac{\prod_i |z_{i\uparrow} - w|^{2\delta_1\alpha}}{\prod_i |z_{i\downarrow} - w|^{2\delta_1\alpha}}, \tag{C6}$$

$$\exp\{i\delta_2\theta(w,w^*)\} \to \frac{\prod_i (z_{i\uparrow} - w)^{\delta_2\alpha}}{\prod_i (z_{i\uparrow}^* - w^*)^{\delta_2\alpha}} \frac{\prod_i (z_{i\downarrow}^* - w^*)^{\delta_2\alpha}}{\prod_i (z_{i\downarrow} - w)^{\delta_2\alpha}}. \tag{C7}$$

(The general formula for the many vertex correlator can be found, for example, in Ref. 33.) The single-valuedness of the WFs demands $\delta_2 = \frac{1}{\sqrt{2}}$. If we take also $\delta_1 = \frac{1}{\sqrt{2}} = \delta_2 = \delta$ then

$$\begin{aligned}
&\langle \exp\{i\delta\phi(w_1,w_1^*)\}\exp\{-i\delta\phi(w_2,w_2^*)\} \\
&\quad\times\exp\{i\delta\theta(w_3,w_3^*)\}\exp\{-i\delta\theta(w_4,w_4^*)\}\rangle \\
&\quad= \frac{1}{|w_1 - w_2|^{2\delta^2}} \frac{1}{|w_3 - w_4|^{2\delta^2}} \\
&\quad\times \frac{(w_1 - w_3)^{\delta^2}}{(w_1^* - w_3^*)^{\delta^2}} \frac{(w_2 - w_4)^{\delta^2}}{(w_2^* - w_4^*)^{\delta^2}} \frac{(w_1^* - w_4^*)^{\delta^2}}{(w_1 - w_4)^{\delta^2}} \frac{(w_2^* - w_3^*)^{\delta^2}}{(w_2 - w_3)^{\delta^2}},
\end{aligned} \tag{C8}$$

and the mutual statistics between any of two particles of different kinds [Eq. (13), (14), (23), and (24)] is fermionic. To see that, for example, for Eq. (13) pair we send 2 toward 4 and switch w_1 and w_3 coordinates.

In our case of the quantum Hall bilayer,

$$\begin{aligned}
\Psi' &= \frac{\prod_{k<l} |z_{k\uparrow} - z_{l\uparrow}|^2 \prod_{p<q} |z_{p\downarrow} - z_{q\downarrow}|^2}{\prod_{i,j} |z_{i\uparrow} - z_{j\downarrow}|^2} \\
&= \mathrm{Det}\left\{\frac{1}{z_{i\uparrow}^* - z_{j\downarrow}^*}\right\} \mathrm{Det}\left\{\frac{1}{z_{k\uparrow} - z_{l\downarrow}}\right\},
\end{aligned} \tag{C9}$$

The same analysis as above will fix $\alpha=1$ and $\delta=\frac{1}{2}$ so that in this case the mutual statistics is semionic just as it should be in the BF CS field theory.

The BF CS theory of a 2D superconductor is[26]

$$\frac{1}{\pi}\epsilon^{\mu\nu\lambda} b_\mu \partial_\nu a_\lambda - a_\mu j^\mu - b_\mu \tilde{j}^\mu, \tag{C10}$$

where a^μ and b^μ, $\mu=0,1,2$ are gauge fields; the first term is the CS term and j^μ and $\tilde{j}^\mu$, $\mu=0,1,2$ represent quasiparticle and vortex density currents. The Lagrangian encodes in the $\frac{1}{\pi}$ coefficient mutual semionic statistics between the two excitations in 2D superconductor—any time quasiparticle encircles vortex it gets the Bohm-Aharonov phase π because vortex corresponds to the half-flux quantum excitation in the paired system. Higher order in derivatives, i.e., Maxwell terms $\sim(\partial \times a)^2$ and $\sim(\partial \times b)^2$ are present in the description of the ordinary (s wave) gapped 2D superconductor and can describe the plasmon modes that are gapped—see Ref. 26. In our case, because from CFT analogies [Eqs. (C6) and (C7)] we find that δ_1 can be continuous and correspond to a branch of gapless excitations, we expect a quadratic in one of the gauge fields, without derivatives, term to describe such a behavior. For example, if we add a term quadratic in b [$b_\mu b^\mu$ with the $(\partial \times a)^2$ Maxwell term present] our classical equations of motion will be: $\Delta a=0$ and $\partial \times b=0$. They describe gapless behavior (Goldstone mode) in one gauge field and associated quasiparticle description, and incompressible behavior in the other.

[The $SU(2)_1 \times \overline{SU(2)_1}$ theory can be described by the following Lagrangian

$$\frac{1}{2\pi}\epsilon^{\mu\nu\lambda} b_\mu \partial_\nu a_\lambda - a_\mu j^\mu - b_\mu \tilde{j}^\mu, \tag{C11}$$

and we see explicitly mutual fermionic statistics.]

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